Developing the amended state space model of the full state observer

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Develop the new state space model for a full order state observer.

Let us assume we wish to design an observer for a system with the following state space model:

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

where the system has an order \( n \), one single input \( u \) and one single output \( y \).

We design an observer with specified poles \( L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{bmatrix} \).

The modified block diagram of the closed loop observer is shown below. Let us assume \( D = 0 \) for now to simplify the derivation.

\[ (Bu + Le) \]

\[ Bu + Ly - LC\hat{x} \]

\[ \hat{x} \]

\[ \hat{y} \]

\[ y - \hat{y} \]

\[ e = y - \hat{y} = y - C\hat{x} \]
We can now think of this system as having 2 inputs \((u \text{ and } y)\) and one output \(y\) and \(n\) states \((\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)\).

So for the new system, the number of inputs \(r = 2\).

\(m = 1\) is still one (one output).

and \(n\) is still the same:

So:

\[
\begin{align*}
\text{Aobs} & \quad n \times n \\
\text{Bobs} & \quad n \times 2 \\
\text{Cobs} & \quad 1 \times n \\
\text{Dobs} & \quad 1 \times 2
\end{align*}
\]

From the block diagram on the previous page:

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu + Ly - LC\hat{x} \\
&= (A - LC)\hat{x} + Bu + Ly
\end{align*}
\]

The input vector is now

\[
\begin{bmatrix}
\vdots \\
u \\
\vdots \\
y
\end{bmatrix}
\]
So we can make Butly more compact:

\[ \text{Butly} = \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} y \end{bmatrix} \]

so the state equation becomes:

\[ \dot{x} = (A-\text{LC})x + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} y \end{bmatrix} \]

So

\[ A_{\text{obs}} = A-\text{LC} \]
\[ B_{\text{obs}} = \begin{bmatrix} B & L \end{bmatrix} \]

Assume \( D = 0 \)

\[ \Rightarrow D_{\text{obs}} = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \]

\[ \hat{y} = C \hat{x} + \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} y \end{bmatrix} \]

\[ C_{\text{obs}} = C \]
\[ D_{\text{obs}} = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \]

The new poles are the eigenvalues of

\[ (A-\text{LC}) \quad \text{i.e.} \quad |S I - A + LC| = 0 \]
Alternative state space model:

An alternative way of looking at the observer model is to consider that the input is the error $e$ rather than the output of the plant $y$. This is shown below in a diagrammatic form:

![Diagram of state space model](image)

$e = y - \hat{y}$

Referring back to the block diagram on page 1 and noting that $e = y - \hat{y}$, we can write the state equations of the observer model as follows:

\[
\dot{x} = A\hat{x} + Bu + Le
\]

Let us think of the new input as a vector:

\[
\begin{bmatrix}
    u \\
    e
\end{bmatrix}
\]

Then we can re-write the equation above as:

\[
\begin{align*}
\dot{x} &= A\hat{x} + \begin{bmatrix} B & L \end{bmatrix}\begin{bmatrix} u \\
    e
\end{bmatrix} \\
\hat{y} &= C\hat{x} + \begin{bmatrix} D & 0 \end{bmatrix}\begin{bmatrix} u \\
    e
\end{bmatrix}
\end{align*}
\]
So
\[ A_{obs} = A \quad n \times n \]
\[ B_{obs} = [B; L] \quad n \times 2 \quad \text{(note: } r = 2) \]
\[ C_{obs} = C \quad 1 \times n \]
\[ D_{obs} = [D; 0] \quad 1 \times 2 \quad (r = 2) \]

Comparison between the poles of controllers and observers:
Controller poles: \[ A_{cont} = A - BK \]
\[ \text{poles } \Rightarrow \left| sI - A + BK \right| = 0 \]
Observer poles: \[ A_{obs} = A - LC \]
\[ \text{poles } \Rightarrow \left| sI - A + LC \right| = 0 \]
Getting the states out of the observer model.

We need to extract the estimated states $\hat{x}$ out of the observer model. In order to do this we need to modify $C$ as shown below.

$$C_{\text{mod}} = \begin{bmatrix} \vdots & C & \vdots \\ n \{ & I & \end{bmatrix}$$

The new number of outputs is $n + 1$ ($1$ is the original output $\hat{y}$; $n$ is necessary to send out the number of states $\hat{x}$).

$$m_{\text{new}} = n + 1$$

Extracting the states is necessary in order to implement a state variable feedback controller (i.e. pole placement).